A Closed-Form Solution for Minimum-Fuel, Constant-Thrust Trajectories

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The transfer of a vehicle in a vacuum between arbitrary boundary conditions with minimum fuel consumption is investigated when two external accelerations (thrust and gravity) are considered. In a previous study, G. W. Cherry assumed that these accelerations could be replaced by a linear function of time. A less severe assumption can be imposed by replacing only the gravity-acceleration terms with polynomials in time. The necessary condition for optimal thrusting for this assumption requires that the primer be a linear function of time. The resulting equations can be integrated completely to yield sufficient equations in terms of the boundary conditions on the state, the five independent constants of the primer vector, and the solution time. Example problems are solved for transfers between fixed states, free terminal argument of periapsis, and free terminal true anomaly and argument of periapsis.

Introduction

THE transfer of a vehicle in a vacuum between two state vectors with minimum-fuel consumption is investigated when two external forces (i.e., those forces caused by thrust and gravity) are considered. If an inverse-square force of attraction and a continuous constant thrust are assumed, the minimization of the fuel (equivalent under these conditions to the minimization of time) requires determining the optimal thrust-direction vector referred to as the primer vector. Therefore, for optimal motion, the direction cosines of the thrust vector are proportional to the components of the primer. The primer and its derivative are related to the state variables by a set of linear differential equations obtained by applying the calculus of variations to the differential equations of motion. Thus, to obtain a closed-form solution to minimum-time constant-thrust trajectories, the linear differential equations that relate the control (primer and primer derivative) to the state must be integrated; then the state equations that use this control must be integrated.

Cherry and Bond[1,2] assumed an acceleration profile that was a polynomial in time. The sum of the gravity and thrust-acceleration vectors is replaced by a vector that is a linear function of time. Therefore, the thrust-acceleration vector is known in terms of the sum of this time-varying vector and the negative of the gravity vector. The form of this control makes integration of the state equations realizable; thus, a closed-form solution of the two-point boundary-value problem is possible. Two disadvantages occur with this solution. First, no assumption is made on how the thrust and the mass vary independently; second, no optimality criterion is imposed. The first disadvantage is overcome somewhat by choosing the solution time and the position vectors such that the thrust is almost constant over the trajectory.

A less severe assumption can be imposed in the state equations by replacing with functions of time only those terms which cannot be integrated directly. In Cartesian coordinates, this term is the gravity-acceleration vector, a function of position and time. For spherical coordinates, this term is a function of position, velocity, and time. The necessary conditions for the thrust direction to be optimal for this system of equations are that the primer derivative be a constant and the primer be a linear function of time.

A similar but more severely restricted solution appears in Ref. 3. Although the assumptions are identical to those assumed here, the problem (formulated in spherical coordinates) requires two further approximations to integrate the state equations and to yield a solution in closed form. These approximations are as follows: 1) the down-range angle or longitude is required to be free; thus, one of the constants of the primer is zero; and 2) a small angle approximation is required in one of the thrust angles. The present formulation has no such restrictions, although the range-free capability is certainly possible.

Problem Definition

Consider a rectangular Cartesian coordinate system. The motion of a vehicle acted upon by only two external forces (thrust and those caused by gravity) is described by the following set of equations:

$$\dot{V} = g + (T/m)l \tag{1}$$

$$\dot{R} = V \tag{2}$$

$$\dot{m} = -T/V_E \tag{3}$$

where R, V, g, and l are the position, velocity, gravity-acceleration, and thrust-direction vectors, respectively. Initially, the gravity-acceleration vector will be considered a function of position and time. A minimum-fuel-consumption problem for which the thrust magnitude T remains a finite constant over the interval of the solution ($t_0 \leq t \leq t_1$) will be considered. The mass flow rate, defined as the negative of the ratio of the thrust to the effective exhaust velocity V_E of the engine, also will be considered constant over the solution time. For these assumptions, a solution for minimum-fuel-consumption is equivalent to a minimum-time problem.

The thrust-direction vector is constrained by

$$l \cdot l = 1 \tag{4}$$

For this system of equations and for the stated assumptions, the necessary conditions for an optimum solution are

$$\dot{P} = -Q \tag{5}$$

$$\dot{Q} = -(P \cdot \nabla)g \tag{6}$$

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$$(T/m)P = 2\epsilon l \tag{7}$$

where P, Q, and ϵ are Lagrange multipliers. (The multipliers P and Q are time-varying, three-dimensional vectors, and ϵ is a scalar.) The vector P, which is the multiplier associated with the velocity vector, has been referred to by Lawden[4] as the primer.

Note from Eq. (7) that the thrust-direction vector is parallel to the primer if the undetermined multiplier ϵ is nonzero. The thrust-acceleration magnitude T/m has already been assumed to have a nonzero value. With this observation and Eq. (4), the necessary condition for the thrust-direction vector to be optimal requires

$$l = P/|P| \tag{8}$$

Lawden,[4] by use of the Weierstrass condition, has shown that Eq. (8) is correct and that the direction of the primer and the thrust vector are identical.

Consider now the gravity-acceleration vector. Note from Eq. (6) that if this vector were not a function of position ($\nabla g = 0$) and hence, only a function of time, the vector

$$\dot{Q} = 0 \tag{9}$$

Consequently, this vector and, thus, the primer can be integrated immediately to yield

$$P = \beta - \alpha t \tag{10}$$

$$Q = \alpha \tag{11}$$

where α and β are three-dimensional constant vectors. If this result is used with Eq. (8), the equations of motion can be written as

$$\dot{V} = g(t) + (T/m)(\beta - \alpha t)/|\beta - \alpha t| \tag{12}$$

$$\dot{R} = V \tag{13}$$

$$\dot{m} = -T/V_E \tag{14}$$

The gravity-acceleration vector in Eq. (12) has been written purposely as a function only of time because the preceding set of equations describe optimal motion under this condition. After $g(t)$ has been defined, this system of equations can be integrated completely (Appendix) to give six equations in terms of the boundary conditions on the state (R,V), the six constants of the vector $P(\alpha,\beta,t)$, and the solution time $(t_1 - t_0)$. The resulting equations are transcendental in the unknowns α, β, and t_1. (For convenience, the initial time t_0 will be zero.) For the number of equations in the solution, apparently too many unknowns exist. However, note from Eq. (12) that the unknowns appear only as a ratio and hence, are not all independent. Thus, the primer will be scaled by a component of the vector β.

To arrive at this solution, the gravity-acceleration vector is required to be only a function of time. Thus, the form of this function and a method of computation must be determined. Let the gravity-acceleration vector be represented by a polynomial in time of the form

$$g(t) = \sum_{n=0}^{j} \Lambda_n t^n \tag{15}$$

where the coefficients Λ_n are three-dimensional vectors, the components of which are constants to be determined. The solution is further restricted to have burning arcs which are small. For an inverse-square force of attraction, the function which is being replaced by the polynomial is equal to

$$g(R,t) = -R/|R|^3 \tag{16}$$

where the equations of motion are nondimensionalized about a gravitational constant of unity. This function for an optimum solution has, at most, two stationary values for the range of burning arcs being considered. Hence, a third-order power series will be sufficient to represent the gravitational acceleration. The following method is used for computing the coefficients. For a solution between fixed states, two values of Eq. (16) are known from the conditions at the boundaries at $t = t_0$ and $t = t_1$. Two more solutions are required to determine the four coefficients of each component of the polynomial. Consider a differential change of Eq. (16)

$$dg = (\partial g/\partial R)dR + (\partial g/\partial t)dt \tag{17}$$

where second and higher order terms have been ignored. If the variation with position is assumed to be zero ($\partial g/\partial R = 0$) over this interval, only the partial with respect to time needs to be computed.

$$\partial g/\partial t = -\partial R/\partial t/|R|^3 + (3R/|R|^4)\partial|R|/\partial t \tag{18a}$$

which may be written as

$$\partial g/\partial t = -V/|R|^3 + 3R(R\cdot V)/|R|^5 \tag{18b}$$

Evaluating Eqs. (17) and (18b) at the two times ($t_0 + dt$ and $t_1 - dt$) will produce the required two additional solutions needed to determine the coefficients of the polynomial in Eq. (15).

Method of Solution

The solution obtained by integrating Eqs. (12–14) between two fixed states is of the form

$$F_i(X_0,X_f,C_j) = 0 \qquad i = 1,2,\ldots 6, j = 1,2,\ldots 6 \tag{19}$$

where X_0 and X_f are the initial and final state vectors, respectively, and each C_j is an undetermined constant of the vectors (α,β) and of the terminal time t_1, only six of which are independent. For a solution between two fixed states, Eq. (19) must be satisfied through the choice of the controls C_j. A cost function may be defined as

$$J = F_i^2 \tag{20}$$

The gradient of J is

$$\nabla J = 2F_j\cdot(\partial F_j/\partial C_i) \tag{21}$$

The problem of finding a solution of Eq. (19) becomes one of minimizing J by some multivariable search method. Because the partial derivatives of F_j with respect to C_i (and hence, the gradient of J) can be computed analytically and are readily available, a minimization technique that uses the gradient components is indicated. One technique that exhibits quadratic convergence characteristics near the minimum is presented in Ref. 5.

Boundary Conditions

Consider now the case for which the solution is constrained to satisfy a specified set of terminal boundary conditions. If the state (V,R) is defined as X and the associated Lagrange multipliers (P,Q) as λ, then from the general transversality condition (Ref. 6), the following equation is obtained at the terminal time

$$\lambda(t_1) = \nu\cdot(\partial M/\partial X)|t_1 \tag{22}$$

where the constraint relations M are of the form

$$M_q[X(t_1),t_1] = 0, \qquad q \leq 6 \tag{23}$$

and each ν is a constant Lagrange multiplier. Eqs. (22) and (23) are $(6 + q)$ equations in terms of the unknowns; the q components of the ν vector and the six components of the terminal state vector. Thus, a sufficient number of equations exist to satisfy the specified terminal constraints. Note that the only effect of adding the terminal constraints [Eq. (23)] to the solution is to increase the dimensions of the function F in Eq. (19).

Example Problems

Fixed State and Free Terminal Argument of Periapsis

Consider the following example problem. At the terminal boundary, the following constraints are to be satisfied

$$M_1 = R\cdot R - R_1^2 = 0,\ M_2 = R\cdot V = 0$$
$$M_3 = V\cdot V - V_1^2 = 0 \tag{24}$$

where R_1 and V_1 are given constants. These constraints, for a two-dimensional problem, require the terminal state to satisfy a given radius magnitude, flight-path angle, and velocity magnitude. The solution that satisfies Eq. (24) is a range-free solution or a solution with free argument of periapsis ω. If Eq. (22) is applied to these constraints and the constant multipliers ν are eliminated, the optimal range-free conduction is

$$y = xu|_{t_1} \tag{25}$$

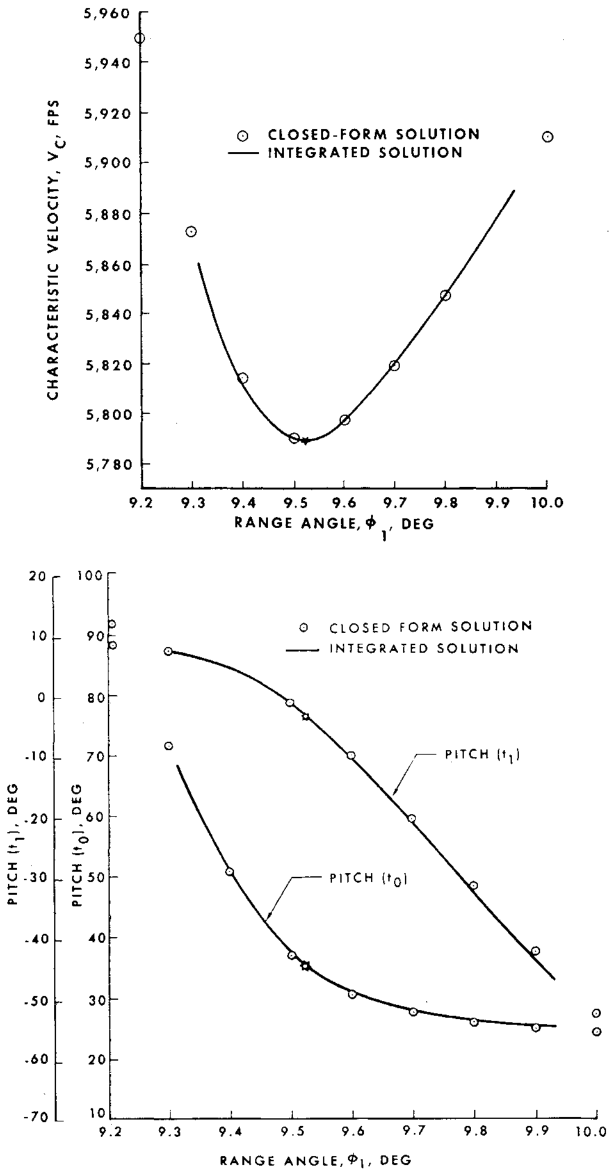

Fig. 1 A comparison of a closed-form and integrated solution as a function of range angle; a) characteristic velocity, b) initial and final pitch angles.

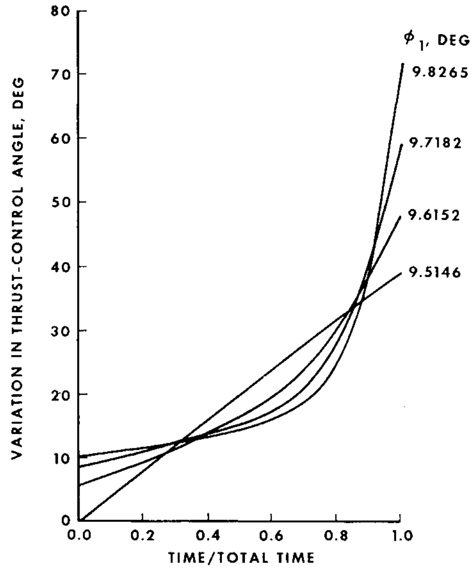

Fig. 2 Variation in the thrust-control angle as a function of time for a number of terminal ranges.

where

$$u = [R_1Q_y - V_1P_x]/[R_1Q_x + V_1P_y] \tag{26}$$

and the subscripts x and y on the vectors P and Q refer to the x and y components. Equations (19, 24, and 25) must be solved simultaneously. An explicit solution for the terminal state may be obtained from a solution of Eqs. (24) and (25)

$$x = KR_1/(1 + u^2)^{1/2},\ y = xu$$
$$V_y = KV_1/(1 + u^2)^{1/2},\ V_x = -V_yu \tag{27}$$

where

$$K = \begin{matrix} +1, x > 0 \\ -1, x < 0 \end{matrix} \tag{28}$$

Thus, for this system of constraints, Eq. (24), the terminal state can be solved for explicity in terms of the controls (α, β, and t_1).

In Fig. 1, a comparison is presented of the closed-form and integrated solutions as depicted by plots of characteristic velocity and initial and final pitch angle (measured from the local horizon) against the terminal range. These solutions are for a launch to lunar orbit, which has an initial thrust-to-weight ratio $T/W_0 = 0.34$, and a specific impulse $I_{sp} = 303$ sec. The integrated solutions were obtained by integrating the nonlinear equations and the associated Euler-Lagrange equations to a terminal cut-off. The corrections to the initial multipliers were obtained by inversion of a difference matrix which was generated by successive integrations up the trajectory. The circles represent solutions obtained by solving Eq. (19) for fixed boundary conditions. The agreement between the two solutions is remarkable. Next Eq. (27) and its partial derivatives with respect to the controls were introduced into Eq. (19). The solution was initiated with a number of terminal-position vectors. In each case, the solution converged to the range-free solution indicated by the star. In Fig. 2, the variation in the thrust-control angle (pitch) is plotted against a nondimensional time for a number of terminal ranges. The control program for a terminal range of $\phi_1 = 9.5146°$ is the range-free solution and varies almost as a linear function of time. The nonlinearity of the control programs as the solutions vary from this range-free case can be noted. Solution time in double-precision arithmetic for any given problem was approximately a few seconds on a Univac 1108 computer.

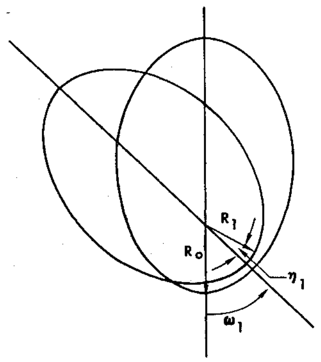

Fig. 3 Initial conditions on two identical ellipses which are misorientated by an angle ω_1.

Free Terminal Argument of Periapsis and True Anomaly

Consider the constraint relations M_q at the terminal time of the form

$$M = R \times V - H = 0$$
$$M_4 = V \cdot V - 2/|R| - E = 0 \tag{29}$$

where H is the angular momentum vector and E is twice the total terminal energy. By application of Eq. (22), the solution for the constant multipliers is found to be

$$\nu_4 = \tfrac{1}{2}[(R \cdot P)/(R \cdot V)] \tag{30a}$$

$$\nu_2 = [y(\nu_4\gamma_z - Q_z) - V_y(P_z - 2\nu_4 V_z)]/H_z \tag{30b}$$

$$\nu_1 = (P_z + \nu_2 x - 2\nu_4 V_z)/y \tag{30c}$$

$$\nu_3 = (\nu_2 z + 2\nu_4 V_x - P_x)/y \tag{30d}$$

where γ is a vector defined as

$$\gamma = (\partial/\partial R)(-2/|R|) \tag{31}$$

Note that the multiplier ν_4 is singular for a terminal state vector on the line of apsides. The two optimal conditions resulting from the application of the constraints given by Eq. (29) are

$$-Q_x - \nu_2 V_z + \nu_3 V_y + \nu_4 \gamma_x = 0 \tag{32a}$$

$$-Q_y + \nu_1 V_z - \nu_3 V_x + \nu_4 \gamma_y = 0 \tag{32b}$$

Equations (19, 29, and 32) comprise a system of 12 equations in terms of the 12 unknowns; the six controls C_j and the six components of the terminal state vector.

In Fig. 3, the initial configuration is illustrated for a transfer problem between two identical ellipses that are misorientated by an angle ω_1. The initial state vector (R_0, V_0) was chosen at the periapsis of the initial ellipse. The final state vector was chosen at a true anomaly of 10° on an ellipse that had an argument of periapsis equal to 45°. A thousand seconds was guessed as the solution time t_1. For a free terminal true anomaly and argument of periapsis, the solution to this problem is obvious; $\omega_1 = \eta_1 = t_1 = 0$. The solution converged to these values is approximately 40 sec on a Univac 1108 computer. Note that the converged solution was indeterminate [ν_4 of Eq. (30)] because the primer and velocity vectors are normal to the radius.

Conclusions

For motion in an inverse square force field and for a constant thrust, the substitution of a polynomial in time for the gravity-acceleration vector in rectangular Cartesian coordinates allows the state and adjoint equations to be integrated in closed form. Solutions to the resulting equations may be obtained by use of a multivariable search technique such as Davidon's.[5] The results obtained agree remarkably well with either known solutions or solutions obtained by classical optimum techniques. For the two example problems presented, the relaxation of the terminal boundary conditions was shown to be a simple and straightforward addition to the solution.

Appendix: Derivation of a Set of Closed-Form Equations to a Minimum-Fuel, Constant-Thrust Solution

The motion of a vehicle in a rectangular Cartesian coordinate system is described by the following set of equations:

$$\dot{V} = \sum_{n=0}^{j} \Lambda_n t^n + \frac{T}{m}\frac{P}{|P|} \tag{A1}$$

$$\dot{R} = V \tag{A2}$$

$$\dot{m} = -T/V_E \tag{A3}$$

where R, V, and P are the position, velocity, and primer vectors, respectively; T is the magnitude of the thrust; V_E is the effective exhaust velocity; and m is the mass of the vehicle. The thrust and exhaust velocity are assumed constant. The necessary conditions for the thrust direction to be optimal for this system of equations require the primer derivative to be a constant and the primer to be a linear function of time.[4] Let P be defined as $P = \beta - \alpha t$, where β and α are three-dimensional vectors defined as

$$\alpha = c_1 l_x + c_2 l_y + c_3 l_z, \; \beta = c_4 l_x + c_5 l_y + c_6 l_z \tag{A4}$$

where l_x, l_y, and l_z are, respectively, unit vectors in the x, y, and z directions, and each c is an undetermined constant. Thus, the primer vector is defined as

$$P = (c_4 - c_1 t)l_x + (c_5 - c_2 t)l_y + (c_6 - c_3 t)l_z \tag{A5}$$

and the absolute value is defined as

$$|P| = \rho \tag{A6}$$

For simplicity, the x component of the position and velocity vectors will be used. The x component of the acceleration vector may be written as

$$\dot{V}_x = \sum_{n=0}^{j} h_{n+1} t^n + \frac{A_0}{\mu}\frac{c_4 - c_1 t}{\rho} \tag{A7}$$

where A_0 is the initial thrust acceleration (T/m_0) and μ is the mass ratio (m/m_0). By multiplication of Equation (A7) by μ and integration between the bounds from $t = t_0$ to $t = t_1$, Eq. (A8) is obtained

$$[\mu V_x - \dot{\mu} x] - \sum_{n=0}^{j} \frac{h_{n+1} t^{n+1}}{n+1}\left[1 + \frac{\dot{\mu} t (n+1)}{n+2}\right]\Bigg|_{t_0}^{t_1} = A_0 \int_{t_0}^{t_1} \frac{(c_4 - c_1 t)dt}{\rho} \tag{A8}$$

The integral I_1 will be defined as

$$I_1 = A_0 \int_{t_0}^{t_1} \frac{(c_4 - c_1 t)dt}{\rho} \tag{A9}$$

The magnitude of the primer can be written in the form

$$\rho = [A + Bt + Ct^2]^{1/2} \tag{A10}$$

where $A = (\beta \cdot \beta)$, $B = -2(\alpha \cdot \beta)$, and $C = (\alpha \cdot \alpha)$. Thus, I_1 is of the form

$$I_1 = A_0 c_4 \int_{t_0}^{t_1} \frac{dt}{[A + Bt + Ct^2]^{1/2}} - A_0 c_1 \int_{t_0}^{t_1} \frac{t\,dt}{[A + Bt + Ct^2]^{1/2}} \tag{A11}$$

and integration formulas 165, 174, and 703 of Peirce[7] may be applied. (Future references to integration formulas will be taken from Peirce.) Therefore

$$I_1 = A_0\left[\frac{c_4}{|\alpha|} - \frac{c_1(\alpha\cdot\beta)}{|\alpha|^3}\right]\ln\frac{M}{N} - \frac{A_0 c_1}{\alpha\cdot\alpha}(\rho_1 - \rho_0) \quad \text{(A12)}$$

where

$$M = |\alpha|\rho_1 - (\alpha\cdot P),\ N = |\alpha|\rho_0 - (\alpha\cdot\beta) \quad \text{(A13)}$$

Thus, the integral of Eq. (A7) between the bounds at $t = 0$ and $t = t_1$

$$F_{V_x} = I_1 + \dot{\mu}(x_1 - x_0) + V_{x_0} - \mu_1 V_{x_1} + \sum_{n=0}^{j}\frac{h_{n+1}t_1^{n+1}}{(n+1)}\left[1 + \frac{\dot{\mu}t_1(n+1)}{n+2}\right] = 0 \quad \text{(A14)}$$

To obtain a second integral of Eq. (A7), the equation must first be integrated as an indefinite integral

$$\mu V_x - \dot{\mu}x - \sum_{n=0}^{j}\frac{h_{n+1}t^{n+1}}{n+1}\left[1 + \frac{\dot{\mu}t(n+1)}{n+2}\right] = A_0 I_2 - H_0 \quad \text{(A15)}$$

where

$$I_2 = \frac{c_4}{C^{1/2}}\sinh^{-1}\left(\frac{2Ct+B}{q^{1/2}}\right) - c_1\left[\frac{\rho}{C} - \frac{B}{2C^{3/2}}\sinh^{-1}\left(\frac{2Ct+B}{q^{1/2}}\right)\right] \quad \text{(A16)}$$

$$q = 4AC - B^2 \quad \text{(A17)}$$

and the constant of integration evaluated at $t = 0$ yields

$$H_0 = \dot{\mu}x_0 - V_{x_0} + A_0\left[\left(\frac{c_4}{C^{1/2}} + \frac{c_1 B}{2C^{3/2}}\right)\times \sinh^{-1}\left(\frac{B}{q^{1/2}}\right) - \frac{c_1\rho_0}{C}\right] \quad \text{(A18)}$$

Equation (A15) is divided by μ^2 to obtain Eq. (A19)

$$\frac{\mu V_x - \dot{\mu}x}{\mu^2} - \sum_{n=0}^{j}\frac{h_{n+1}t^{n+1}}{\mu^2(n+1)}\times\left[1 + \frac{\dot{\mu}t(n+1)}{n+2}\right] = \frac{A_0 I_2 - H_0}{\mu^2} \quad \text{(A19)}$$

Successive applications of formula 67 to the summation term or noting by inspection the exact differential in Eq. (A19) yields

$$\frac{d}{dt}\left[\frac{x}{\mu} - \sum_{n=0}^{j}\frac{h_{n+1}t^{n+2}}{\mu(n+1)(n+2)}\right] = \frac{A_0 I_2 - H_0}{\mu^2} \quad \text{(A20)}$$

Integration of the left side between bounds $t = 0$ and $t = t_1$ yields

$$\frac{x_1}{\mu_1} - x_0 - \sum_{n=0}^{j}\frac{h_{n+1}t_1^{n+2}}{\mu_1(n+1)(n+2)} \quad \text{(A21)}$$

The second term on the right of Eq. (A20) is

$$-H_0\int_{t_0}^{t_1}\frac{dt}{\mu^2} = \frac{-H_0 t_1}{\mu_1} \quad \text{(A22)}$$

The first term on the right of Eq. (A20) is

$$A_0\int_{t_0}^{t_1}\frac{I_2 dt}{\mu^2} = A_0\int_{t_0}^{t_1}\frac{I_2 dt}{(1+\dot{\mu}t)^2} = A_0\left[\frac{c_4}{C^{1/2}} + \frac{c_1 B}{2C^{3/2}}\right]\int_{t_0}^{t_1}\frac{\sinh^{-1}K dt}{(1+\dot{\mu}t)^2} - \frac{A_0 c_1}{C}\int_{t_0}^{t_1}\frac{\rho dt}{\mu^2} \quad \text{(A23)}$$

where

$$K = (2Ct + B)/q^{1/2} \quad \text{(A24)}$$

Let the integral S_1 be defined as

$$S_1 = A_0\int_{t_0}^{t_1}\frac{\sinh^{-1}K dt}{(1+\dot{\mu}t)^2} \quad \text{(A25)}$$

Integration of S_1 by parts yields

$$S_1 = \frac{-A_0}{\mu\dot{\mu}}\ln\left[\frac{2Ct + B + (A + 4Bt + 4Ct^2)^{1/2}C^{1/2}}{q^{1/2}}\right] + \frac{A_0 C^{1/2}}{\dot{\mu}k^{1/2}}\ln\left[\frac{2k + \mu\sigma - 2\dot{\mu}k^{1/2}\rho}{\mu}\right]\Bigg|_{t_0}^{t_1} \quad \text{(A26)}$$

(formulas 703 and 200, respectively) where

$$k = \gamma\cdot\gamma,\ \sigma = -2(\alpha\cdot\gamma),\ \gamma = \alpha + \dot{\mu}\beta \quad \text{(A27)}$$

If Eq. (A26) is evaluated between the bounds indicated, the integral of S_1 may be written

$$S_1 = (A_0/\dot{\mu})[\ln N/Q - (1/\mu_1)\ln M/Q + (|\alpha|/|\gamma|)\ln J/L] \quad \text{(A28)}$$

where

$$Q = [(\beta\cdot\beta)(\alpha\cdot\alpha) - (\alpha\cdot\beta)^2]^{1/2} \quad \text{(A29a)}$$

$$J = (\gamma\cdot\gamma) - (\alpha\cdot\gamma)\mu_1 - \dot{\mu}\rho_1|\gamma| \quad \text{(A29b)}$$

$$L = \mu_1[(\gamma\cdot\gamma) - (\alpha\cdot\gamma) - \dot{\mu}\rho_0|\gamma|] \quad \text{(A29c)}$$

Let the integral S_2 be defined as

$$S_2 = A_0\int_{t_0}^{t_1}\frac{\rho dt}{\mu^2} \quad \text{(A30)}$$

Application of formula 212b yields

$$S_2 = \frac{A_0}{\dot{\mu}}\left[\left(\rho_0 - \frac{\rho_1}{\mu_1}\right) - \frac{\alpha\cdot\gamma}{|\gamma|}\ln\frac{J}{L} + \frac{|\alpha|}{\dot{\mu}}\ln\frac{M}{N}\right] \quad \text{(A31)}$$

Therefore, the integral of the first term on the right side of Eq. (A20) is

$$A_0\int_{t_0}^{t_1}\frac{I_2\, dt}{\mu^2} = \left[\frac{\alpha\cdot(c_4\alpha - c_1\beta)}{|\alpha|^3}\right]S_1 - \frac{c_1 S_2}{|\alpha|^2} \quad \text{(A32)}$$

The integral of Eq. (A20) may be written

$$F_x = x_0 - \frac{x_1}{\mu_1} + \sum_{n=0}^{j}\frac{h_{n+1}t_1^{n+2}}{\mu_1(n+1)(n+2)} - \frac{H_0 t_1}{\mu_1} + \left[\frac{\alpha\cdot(c_4\alpha - c_1\beta)}{|\alpha|^3}\right]S_1 - \frac{c_1 S_2}{|\alpha|^2} = 0 \quad \text{(A33)}$$

The integrals of $\dot{v}_y$ and $\dot{v}_z$ may be written directly from the integration process of $\dot{v}_x$, changing only the constants $c_1, \ldots, c_6$, and the position and velocity component. Now redefine I_1 and H_0 in general terms and rewrite Eqs. (A14) and (A33) to encompass the integrations of $\dot{v}_y$ and $\dot{v}_z$

$$F_V = I_V + \dot{\mu}(R_1 - R_0) + V_0 - \mu_1 V_1 + \sum_{h=0}^{j}\frac{\Lambda_{n+1}t_1^{n+1}}{n+1}\left[1 + \frac{\dot{\mu}t_1(n+1)}{n+2}\right] = 0 \quad \text{(A34)}$$

$$F_R = R_0 - \frac{R_1}{\mu_1} + \sum_{n=0}^{j}\frac{\Lambda_{n+1}t_1^{n+2}}{\mu_1(n+1)(n+2)} - \frac{H_R t_1}{\mu_1} + \left[\frac{\alpha\cdot(c_j\alpha - c_i\beta)}{|\alpha|^3}\right]S_1 - \frac{c_i S_2}{|\alpha|^2} = 0 \quad \text{(A35)}$$

where

$$I_V = A_0\left[\frac{c_j}{|\alpha|} - \frac{c_i(\alpha\cdot\beta)}{|\alpha|^3}\right]\ln\frac{M}{N} - \frac{A_0 c_i}{|\alpha|^2}(\rho_1 - \rho_0) \quad \text{(A36)}$$

and

$$H_R = A_0\left[\frac{c_j}{|\alpha|} - \frac{c_i(\alpha\cdot\beta)}{|\alpha|^3}\right]\sinh^{-1}\left[-\frac{\alpha\cdot\beta}{Q}\right] - \frac{A_0 c_i\rho_0}{\alpha\cdot\alpha} + \dot{\mu}R_0 - V_0 \quad \text{(A37)}$$

The components and subscripts of the vector Eqs. (A34) and (A35) may be obtained by the following relationship:

i	R	V	Λ	j
1	x	V_x	h	4
2	y	V_y	a	5
3	z	V_z	b	6

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Vibration Modes of Large Structures by an Automatic Matrix-Reduction Method

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An automatic matrix reduction method is presented whereby the lower modes of structural systems with many degrees-of-freedom can be extracted by solving an eigenvalue problem of much smaller size than the actual inertia matrix. The process is effected without arbitrary lumping of masses at judiciously selected physical node-points. The method is based upon Crandall's tailoring of an eigenvalue routine created by Lanczos. The present work corrects the basic weakness in the original method, which is its numerical instability. In addition, practical suggestions for the method's implementation are made and some empirically-based conclusions are drawn concerning the relationship between the exact and reduced frequency spectrums.

Nomenclature

$[A],[B]$	= positive definite and symmetric matrices ($n \times n$)
$\{C\}$	= modal components of starting vector $\{v_1\}$
c_i	= element of $\{C\}$ vector
$[I]$	= identity matrix
$[K]$	= assembled stiffness matrix ($n \times n$)
$[M]$	= assembled mass matrix ($n \times n$)
m	= number of reduced degrees-of-freedom
n	= number of original unreduced degrees-of-freedom
$[V]$	= transformation matrix used to effect reduction from n to m degrees-of-freedom ($n \times m$)
$\{v_i\}$	= Lanczos vectors
$\{v_i{}^*\}$	= see Eq. (10)
$\{\bar{v}_i\}$	= see Eq. (26)
$\{v_i{}^{(s)}\}$	= see Eq. (27)
$[X]$	= modal matrix for original system ($n \times n$)
$\{x\}$	= eigenvector of original system
$\{y\}$	= generalized coordinate of reduced $m \times m$ system
α_i	= see Eqs. (9) and (20)
β_i	= see Eqs. (13) and (18)
$\gamma_{i,j}$	= see Appendix
δ_{ij}	= Kronecker delta; equals $\begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$
λ	= eigenvalue of unreduced system
λ_R	= Rayleigh quotient and eigenvalue of reduced system
$\lceil \lambda \rfloor$	= diagonal eigenvalue matrix of unreduced system
μ_i	= see Eq. (8)
ω	= natural frequency of unreduced system
$[\ ]^T$	= denotes matrix transpose
$\{\ \}^T$	= denotes vector transpose or row vector

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Introduction

STATIC, elastic stress analyses by computerized finite-element methods have evolved to a high level of sophistication and accuracy[1,2] and are now accepted as necessary tools in the analysis of extremely complex structures. The